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Super-critical withdrawal from a two-layer fluid through a line sink if the lower layer is of finite depth

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The steady response of a fluid consisting of two regions of different density, the lower of which is of finite depth, is considered during withdrawal. Super-critical flows are considered in which water from both layers is being withdrawn, meaning that the interface is drawn down directly into the sink. The results indicate that if the flow rate is above some minimum, the angle of entry of the interface depends more strongly on the relative depth of the sink than on the flow rate. This has quite dramatic consequences for the understanding of selective withdrawal from layered fluids.

1. Introduction

It has long been known that when water is drawn from a stratified fluid there is a vertical withdrawal zone from which the bulk of the water flows out through the orifice. The reason for this zone is that less-dense water above the level of the sink must act against its buoyancy to reach the outlet and if it is more than a certain distance away the withdrawal force is not sufficient to overcome this. Similarly, the more-dense water beneath the level of the outlet must overcome gravity to reach the outlet. If the stratification is manifested as layers of different density, rather than some reasonably continuous density variation, the bounds of the layer nearest to the sink act as the boundaries of the withdrawal zone if the effective flow rate is less than some critical parameter. If this parameter exceeds the critical value, then the layer on the other side of the density interface will also flow into the sink.

The importance of understanding this behaviour in managing water quality in storage reservoirs, power station cooling ponds and solar ponds is clear (see Imberger & Hamblin 1982), yet there remain unanswered questions. In particular, the nature and value of this critical flow parameter is still subject to some conjecture, both in the case of two-dimensional flows (into a line sink) and three-dimensional flows (point sink). A great deal of effort has been expended in examining this issue, both experimentally and theoretically, but in the case of discrete layers of different density, the results have been slightly ambiguous. It has always been assumed that the critical parameter was the flow rate. Experimentally, there has been a wide variation in the values obtained for the critical flow rate. Many factors such as the thickness of the interface, the size of the outlet, and reflections of the transient wave from the startup have been suggested as a possible cause of this variation. Various theoretical arguments have been produced to justify these results, but it is still doubtful if any of the results could be said to be definitive.

Numerical investigations have had even more difficulty in coming to a conclusion about the critical flow rate and the behaviour of the flow in the subcritical regime. There exist a multitude of papers discussing withdrawal flow through a line sink from a single layer of fluid beneath a free surface, or beneath an interface between a flowing layer and a stagnant layer. The most relevant of these examine what have come to be known as cusp solutions (see for example Craya 1949; Tuck & Vanden-Broeck 1984; Hocking 1985; Vanden-Broeck & Keller 1987; King & Bloor 1988, Hocking 1988, 1991*a*), in which the interface is drawn down into a cusp shape directly above the sink. Hocking (1995) showed that these solutions are characteristic of the transition between the single-layer flow and the supercritical two-layer flow for the case in which the lower layer is of infinite extent. In that situation, the cusp solutions occur at a unique flow rate for a given geometry. That this might be the case in a fluid in which the lower layer is of finite depth is not so clear because cusp solutions have been found to exist for a range of different flow rates.

Forbes & Hocking (1998) showed that in a duct containing two layers, solutions involving flow from both layers occurred over a continuum of Froude numbers and sink heights provided the Froude number was greater than one. Only a single branch of fully two-layer flows, in which the upstream depth could not be specified *a priori*, was found for Froude numbers less than unity.

Experimental work (Gariel 1949; Harleman & Elder 1965; Jirka 1979; Hocking 1991*b*) found a large scatter in the values of the critical flow rate over a range of different geometries, and a theoretical investigation of the flow is important in interpreting results obtained in experiments and in the field. Of all of this work, only the experiments of Gariel (1949) involved a line sink situated off the bottom of the channel. Solution of the Navier–Stokes equations in such problems using finite-difference or finite-element techniques is fraught with difficulties because of the moving interface, and the very rapid transitions in density and velocity within.

Most of the analytical and numerical work has been undertaken using a model of the flow as steady and irrotational and the fluid as inviscid and incompressible. Much of this work has involved subcritical flows in the hope that this would reveal a limiting solution which would indicate the critical transition. However, in most cases this has not eventuated, and although some useful results have been obtained, the behaviour near and at the critical flow has remained elusive. This model of the flow clearly neglects some features which may be of importance in determining the exact details of the flow in a real situation, such as viscous effects near the walls and along the interface, the time-dependent nature of the flow as the level falls, and the transient effects of opening the outlet.

In this paper, this ideal fluid model is extended to consider the flow when both layers are being drawn through the slot. The lower layer is assumed to be of finite depth, and the upper to be unbounded. The interface is assumed to be very thin, and once again steady, irrotational flow of an incompressible, inviscid fluid is considered. This paper is the third in a series investigating two-layer flows using this model. The first (Hocking 1995) considered the flow into a line sink from two unbounded layers of different density separated by an infinitesimal interface, while the second (Forbes & Hocking 1998) considered a two-layer flow confined by impermeable boundaries both above and below, i.e. flow in a duct.

The problem can be shown to be characterized by three main non-dimensional quantities. The first is the Froude number in the lower layer,

$$F_B = \left(\frac{U_1^2}{g' H_B} \right)^{1/2}, \tag{1.1}$$

where U_1 is the upstream velocity in the lower layer, $g' = (\Delta\rho/\rho)g$ is the effective gravity, where g is the acceleration due to gravity, $\Delta\rho$ is the density difference between the two layers, ρ is a reference density, and H_B is the depth of the lower layer beneath the upstream level of the interface. A second parameter of importance is the height of the sink relative to the depth of the lower layer of the fluid, i.e. $\delta = H_S/H_B$. Finally, the angle of entry, α, of the interface into the sink is of importance. We will show that the angle of entry, α, is very strongly dependent on the value of the relative sink height δ, and only weakly dependent on the value of the Froude number. It will also be shown that solutions of this type only appear to occur for values of Froude number greater than unity except for a small group with values of $\delta \approx 1$–1.5 .

In an experimental situation, this lower-layer Froude number is not easy to measure. However, it is possible to define a total Froude number based on the total mass flux and the height of the sink off the bottom. Thus,

$$F_{\text{Tot}} = F_B \delta^{-3/2} \left[1 + \gamma^{1/2} \left(\frac{\pi - 2\alpha}{\pi + 2\alpha} \right) \right], \tag{1.2}$$

where $\gamma = \rho_2/\rho_1$ is the density ratio between the layers. During an experiment we expect that this quantity would remain approximately constant.

Previous work on this problem has been divided into theoretical work in the subcritical regime, in which the flow is restricted to the lower (high-density) region of the fluid, and experimental work. It is interesting that if one assumes that the flow in the upper layer is stagnant, the equations describing the flow of a single layer of fluid beneath a free surface are identical to those describing the flow in the lower layer of a two-layer fluid beneath an interface, except that the gravity g is replaced by the effective gravity g' (see e.g. Jirka 1979; Yih 1980).

An exact solution in the sub-critical regime was found by Sautreaux (1901), and subsequently by Craya (1949). This solution consists of a downward cusp in the interface directly above the sink, and occurs when there is a wall sloping downward from the sink with an angle of 30° to the horizontal. Tuck & Vanden-Broeck (1984) used a numerical series truncation method to compute a similar cusped flow for the case of a line sink in a single fluid region confined only by a free surface. Further numerical solutions of this type were later obtained by Hocking (1985) and Vanden-Broeck & Keller (1987) for slightly different geometries, including some in which the lower layer was of finite depth. One interesting aspect of the solutions in which the fluid domain is not of finite depth is that the cusped solutions occur at a unique value of the Froude number, while in the finite depth problem, such cusped solutions occur over a continuous range of Froude numbers (Vanden-Broeck & Keller 1987; Hocking 1991*a*). However, Vanden-Broeck & Keller (1987) found an isolated branch of solution that extended down into the region in parameter space where the Froude number is less than one. This has as a limiting case the cusp solution in a fluid of infinite depth computed by Tuck & Vanden-Broeck (1984).

In the case of withdrawal through a thin slot or line sink, i.e. the two-dimensional problem, the experimental work is restricted to that of Gariel (1949), Harleman & Elder (1965), Wood & Lai (1972) and Hocking (1991*b*). In almost all of these cases it was found that the critical drawdown point, at which the upper fluid begins to flow

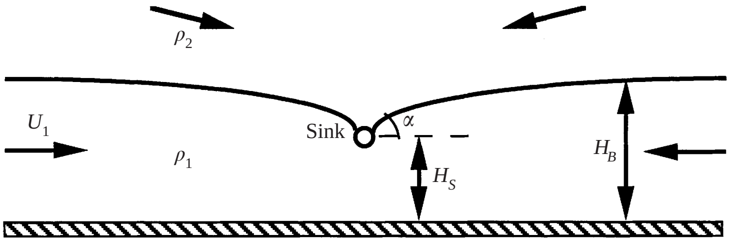

FIGURE 1. Definition sketch.

out through the sink, occurs at a Froude number much lower than that predicted by the cusped solutions outlined above. This result is consistent with the experimental results for a point sink, in which the drawdown occurs at much lower values of the Froude number than expected (Harleman & Elder 1965; Jirka & Katavola 1979; Lawrence & Imberger 1979).

Section 2 describes the formulation of the problem and § 3 details the numerical scheme that is used to solve the resulting system of equations. Section 4 presents the results and § 5 discusses their implications.

It will be shown that there is a very strong relationship between the angle of entry into the sink and the value of the sink height. The angle of entry will be shown to depend only weakly on the Froude number. As the angle of entry of the interface approaches 90°, the flow nears a single-layer flow, and we can make some inferences about the critical flow parameters. It has always been thought that for a fixed sink depth, the Froude number could be increased until some critical value at which the drawdown would occur. At higher values of the Froude number, it was then assumed that the angle of entry would decrease as F_B increased. The results presented in this paper suggest very strongly that this is not the case, and that the determining factor is the relative sink height. These results are qualitatively very similar to the results for a two-layer flow in a duct (Forbes & Hocking 1998).

2. Problem formulation

The steady, irrotational motion of an inviscid, incompressible fluid in two dimensions is to be examined. The fluid is separated by an interface of infinitesimal thickness into two homogeneous regions of different density. The lower of the two regions is assumed to be of finite depth, and the point of withdrawal, in this case a line sink, is set at some height H_S above the bottom. The solutions we seek are those in which the interface is drawn down (or drawn up if the sink height is above the upstream level of the interface) to a point where it enters the sink with an angle α to the horizontal. Fluid is being withdrawn from both above and below the interface (see figure 1).

Let $z = x + \mathrm{i}y$ be the physical plane, with the origin directly beneath the line sink on the bottom of the lower layer. Let the height of the line sink be H_S, and the height of the interface a long way from the sink be H_B. If $y = \eta(x)$ is the equation of the interface, suppose the region below the interface to have density ρ_1, and the region above the interface to have density ρ_2. The velocity potentials of the separate flow fields below and above the interface must both satisfy Laplace's equation, i.e.

$$\left.\begin{aligned} \nabla^2 \Phi_1(x, y) = 0, \quad & 0 < y < \eta(x), \quad x > 0, \\ \nabla^2 \Phi_2(x, y) = 0, \quad & \eta(x) < y < \infty, \quad x > 0. \end{aligned}\right\} \tag{2.1}$$

As the sink is approached, the velocity potentials must have the correct behaviour for a line sink, which is

$$\left.\begin{aligned} \Phi_1 &\to -\frac{m_1}{2\pi}\log\sqrt{x^2+(y-H_S)^2} \quad \text{as } (x,y)\to(0,H_S),\ \ 0<y<\eta(x) \\ \Phi_2 &\to -\frac{m_2}{2\pi}\log\sqrt{x^2+(y-H_S)^2} \quad \text{as } (x,y)\to(0,H_S),\ \ y>\eta(x), \end{aligned}\right\} \tag{2.2}$$

where m_1 and m_2 are the effective respective sink strengths within the two regions. There is a relationship between these two values which must hold if the dynamic condition on the interface is to be satisfied. Applying the Bernoulli equation to the streamline along the interface, and noting that for steady flow there must be no pressure difference across the interface leads to the result that

$$\begin{aligned} &\rho_1 g(\eta(x)-H_B)+\tfrac{1}{2}\rho_1(\Phi_{1x}^2+\Phi_{1y}^2-U_1^2) \\ &\qquad = \rho_2 g(\eta(x)-H_B)+\tfrac{1}{2}\rho_2(\Phi_{2x}^2+\Phi_{2y}^2) \quad \text{on } y=\eta(x), \end{aligned} \tag{2.3}$$

where U_1 is the horizontal velocity upstream in the lower layer. This equation can be rearranged to give

$$2g'(\eta(x)-H_B)+((\Phi_{1x}^2+\Phi_{1y}^2-U_1^2)-\gamma(\Phi_{2x}^2+\Phi_{2y}^2))=0, \tag{2.4}$$

where $g'=((\rho_1-\rho_2)/\rho_1)g$, and $\gamma=\rho_2/\rho_1$.

We note in passing that if the velocity in the upper layer is zero (stagnant fluid), (2.4) becomes

$$2g'(\eta(x)-H_B)+(\Phi_{1x}^2+\Phi_{1y}^2-U_1^2)=0,$$

which is identical to the equation for constant pressure on a free surface, except that g is replaced by g'. Therefore the work done in solving free-surface flow problems relates directly to the present two-layer flow situation, and is an analogue of the single-layer flow behaviour.

Considering the behaviour of the flow near the sink (2.2), we see that in order to satisfy (2.4), it is necessary that

$$m_1=\gamma^{1/2}m_2. \tag{2.5}$$

This result seems slightly surprising at first, but Hocking (1995) showed how it is consistent with considering the flow as an approximation to withdrawal through a thin horizontal slot which extends to $x=-\infty$. It was shown that there is a relationship between the angle of entry into the sink and the depth of the interface downstream in the withdrawal slot. Huber (1960) was able to show that if the two outer layers are of finite depth, there is a direct relationship between the Froude number and the angle of entry. However, in the unconfined flow considered here, there is no way to determine this relationship between Froude number and angle of entry without solving the full system of equations.

The final conditions to be satisfied are that there be no flow across the interface or the solid bottom boundary to the lower layer. The first of these is satisfied provided

$$\eta'(x)=\frac{\Phi_{1y}}{\Phi_{1x}}=\frac{\Phi_{2y}}{\Phi_{2x}} \quad \text{on } y=\eta(x), \tag{2.6}$$

and the second is satisfied provided

$$\nabla\Phi_1\cdot\boldsymbol{n}=0 \quad \text{on } y=0, \tag{2.7}$$

where $\boldsymbol{n}$ is the outward normal to the bottom boundary.

The number of parameters in the problem is reduced if we non-dimensionalize with respect to the sink height H_S as the length scale, and use different velocity scales in the upper and lower regions, so that we have $\hat{y} = y/H_S$, $\hat{x} = x/H_S$, $\hat{q}_1 = (m_1/2\pi H_S)q_1$ and $\hat{q}_2 = (m_1\gamma^{1/2}/2\pi H_S)q_2$, where q_1 and q_2 are the fluid speeds in the two regions. The non-dimensional form of the dynamic condition becomes

$$8\pi^2 Q^{-2}(\hat{\eta} - \lambda) + ((\hat{\Phi}_{1\hat{x}})^2 + (\hat{\Phi}_{1\hat{y}})^2) - ((\hat{\Phi}_{2\hat{x}})^2 + (\hat{\Phi}_{2\hat{y}})^2) = \left(\frac{\pi/2 + \alpha}{\lambda}\right)^2, \tag{2.8}$$

where

$$Q = \left(\frac{m_1^2}{g' H_S^3}\right)^{1/2} \tag{2.9}$$

is a non-dimensional parameter related to the Froude number, λ is the upstream height of the interface, so that $1/\lambda = H_S/H_B = \delta$ and the term on the right-hand side is the square of the non-dimensional upstream velocity in the lower layer. Note that $\pi/2 + \alpha$ is the flux into this lower layer. The line sink is situated at $y = 1$ in this system.

In non-dimensional variables, the behaviour as the sink is approached is given by

$$\left.\begin{aligned} \hat{\Phi}_1 &\to -\log\sqrt{\hat{x}^2 + (\hat{y} - 1)^2} \quad \text{as } (\hat{x}, \hat{y}) \to (0, 1),\ \ 0 < \hat{y} < \hat{\eta}(\hat{x}) \\ \hat{\Phi}_2 &\to -\log\sqrt{\hat{x}^2 + (\hat{y} - 1)^2} \quad \text{as } (\hat{x}, \hat{y}) \to (0, 1),\ \ \hat{\eta}(\hat{x}) < \hat{y} < \infty. \end{aligned}\right\} \tag{2.10}$$

We will henceforth dispense with the 'hat' notation for dimensionless variables.

We solve this problem numerically using a variation on the standard boundary element method (see e.g. Brebbia 1978; Liggett & Liu 1983; Forbes 1985), by considering two analytic functions (one for each region) which exclude the sink-like behaviour. Suppose the complex potentials are given by

$$f_k(z) = \Phi_k + \mathrm{i}\Psi_k, \quad k = 1, 2, \tag{2.11}$$

where $\Psi_k, k = 1, 2$, are the streamfunctions in the two regions. The derivative of the complex potential in each region gives the horizontal and vertical components of velocity $(u_k, v_k), k = 1, 2$, respectively as

$$f_k'(z) = u_k - \mathrm{i}v_k, \quad k = 1, 2. \tag{2.12}$$

We define two functions, $\chi_1(z), \chi_2(z)$ so that

$$\left.\begin{aligned} f_1'(z) &= -\frac{1}{z - \mathrm{i}} - \frac{1}{z + \mathrm{i}} + \chi_1(z), \\ f_2'(z) &= -\frac{1}{z - \mathrm{i}} + \chi_2(z). \end{aligned}\right\} \tag{2.13}$$

The term $-(z - \mathrm{i})^{-1}$ thus builds in the sink behaviour as $z \to \mathrm{i}$, and the extra term $-(z + \mathrm{i})^{-1}$ in the equation for f_1' , combined with an image interface at $y = -\eta(x)$ enforces the condition of no flow through the bottom boundary, given by (2.7), by symmetry.

The functions $\chi_k(z) = A_k - \mathrm{i}B_k, k = 1, 2$, must be analytic in their respective regions, and we can use Cauchy's integral formula to derive integral equations for these quantities, which when combined with the other conditions will provide a complete formulation of the problem. Following Forbes (1985), we apply Cauchy's integral

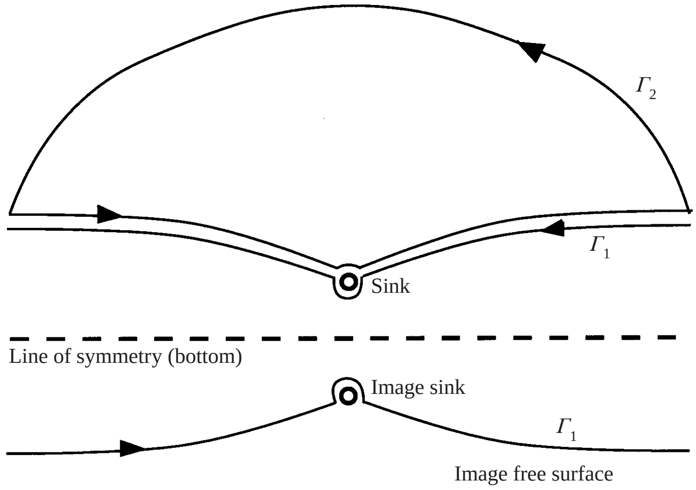

FIGURE 2. Contours used in derivation of the integral equations (2.19).

formula to $\chi_k, k = 1, 2$, on the regions below and above the interface, to get

$$\pi\chi_1(z_0) = \int_{\Gamma_1} \frac{\chi_1(z)}{z - z_0}\,\mathrm{d}z, \quad \pi\chi_2(z_0) = \int_{\Gamma_2} \frac{\chi_2(z)}{z - z_0}\,\mathrm{d}z, \tag{2.14}$$

where Γ_1 and Γ_2 are the contours shown in figure 2, and z_0 lies on the boundary in each case. Now since $\chi_2(z) \to 0$ as $|z| \to \infty$, the contribution of that part of Γ_2 which consists of the circular arc can be shown to be zero. Thus we only need to integrate along the interface. The choice of Γ_1 with an image sink and free surface at $y = -1$ and $y = \eta(x)$, respectively, ensures that there is no flow through $y = 0$. We will later use symmetry to reduce this integral to be only along the actual free surface.

Let s be the arclength along the interface starting from the sink, so that

$$\left(\frac{\mathrm{d}x}{\mathrm{d}s}\right)^2 + \left(\frac{\mathrm{d}\eta}{\mathrm{d}s}\right)^2 = 1. \tag{2.15}$$

Using this as the variable of integration and applying the Chain rule we obtain

$$\pi\mathrm{i}\chi_1(z(s)) = ⨍_{-\infty}^{\infty} \frac{\chi_1(z(t))\,\mathrm{d}z/\mathrm{d}t}{z(t) - z(s)}\,\mathrm{d}t - \int_{-\infty}^{\infty} \frac{\chi_1(z(t_I))\,\mathrm{d}z/\mathrm{d}t_I}{z(t_I) - z(s)}\,\mathrm{d}t_I \tag{2.16a}$$

(where the subscript I refers to the image surface at $y = -\eta(x)$) and

$$-\pi\mathrm{i}\chi_2(z(s)) = ⨍_{-\infty}^{\infty} \frac{\chi_2(z(t))\,\mathrm{d}z/\mathrm{d}t}{z(t) - z(s)}\,\mathrm{d}t. \tag{2.16b}$$

Since A_k and $B_k, k = 1, 2$, can be related to each other along the interface using the condition (2.6), these represent two integral equations for $A_k, k = 1, 2$, respectively. Taking the real parts, and utilizing the symmetry of the situation about the line $x = 0$, i.e.

$$\left.\begin{aligned} &x(-s) = -x(s), \quad y(-s) = y(s), \\ &x'(-s) = x'(s), \quad y'(-s) = -y'(s), \\ &u_k(-s) = -u_k(s), \quad v_k(-s) = v_k(s), \quad k = 1, 2, \end{aligned}\right\} \tag{2.17}$$

and also about $y=0$ for the lower fluid only,

$$\left.\begin{aligned} x_I(s) &= x(s), \quad y_I(s) = -y(s), \\ x_I'(s) &= x'(s), \quad y_I'(s) = -y'(s), \\ u_{I1}(s) &= u_1(s), \quad v_{I1}(s) = -v_1(s), \end{aligned}\right\} \tag{2.18}$$

where the subscript I refers to the image free surface, the integral equations become

$$\begin{aligned} \pi A_1(s) = \int_0^\infty & A_1(t)\left[\frac{x'(t)\Delta y - y'(t)\Delta x}{\Delta x^2+\Delta y^2} - \frac{x'(t)\Delta y - y'(t)\Delta x_+}{\Delta x_+^2+\Delta y^2}\right] \\ & + A_1(t)\left[\frac{x'(t)\Delta y_+ - y'(t)\Delta x}{\Delta x^2+\Delta y_+^2} - \frac{x'(t)\Delta y_+ - y'(t)\Delta x_+}{\Delta x_+^2+\Delta y_+^2}\right] \\ & + B_1(t)\left[\frac{x'(t)\Delta x + y'(t)\Delta y}{\Delta x^2+\Delta y^2} - \frac{x'(t)\Delta x_+ + y'(t)\Delta y}{\Delta x_+^2+\Delta y^2}\right] \\ & + B_1(t)\left[\frac{x'(t)\Delta x + y'(t)\Delta y_+}{\Delta x^2+\Delta y_+^2} - \frac{x'(t)\Delta x_+ + y'(t)\Delta y_+}{\Delta x_+^2+\Delta y_+^2}\right] \mathrm{d}t, \end{aligned} \tag{2.19a}$$

and

$$\begin{aligned} \pi A_2(s) = -\int_0^\infty & A_2(t)\left[\frac{x'(t)\Delta y - y'(t)\Delta x}{\Delta x^2+\Delta y^2} - \frac{x'(t)\Delta y - y'(t)\Delta x_+}{\Delta x_+^2+\Delta y^2}\right] \\ & + B_2(t)\left[\frac{x'(t)\Delta x + y'(t)\Delta y}{\Delta x^2+\Delta y^2} - \frac{x'(t)\Delta x_+ + y'(t)\Delta y}{\Delta x_+^2+\Delta y^2}\right] \mathrm{d}t, \end{aligned} \tag{2.19b}$$

where $\Delta x = x(t) - x(s)$, $\Delta x_+ = x(t) + x(s)$, $\Delta y = y(t) - y(s)$, and $\Delta y_+ = y(t) + y(s)$. Coupled with these two equations we have from (2.7) and (2.8) that

$$B_1(t) = \eta'(x)\left[A_1(t) - \frac{x}{x^2+(y-1)^2} - \frac{x}{x^2+(y+1)^2}\right] + \frac{y-1}{x^2+(y-1)^2} + \frac{y+1}{x^2+(y+1)^2}, \tag{2.20a}$$

and in the upper region

$$B_2(t) = \eta'(x)\left[A_2(t) - \frac{x}{x^2+(y-1)^2}\right] + \frac{y-1}{x^2+(y-1)^2}, \tag{2.20b}$$

on the interface, and therefore equations (2.19) constitute integral equations for A_1 and A_2 along the unknown interface. Using the arclength formulation, the interface condition (2.6) simplifies a little to become

$$8\pi^2 Q^{-2}(\eta(s) - \lambda) + \Phi_{1s}^2 - \Phi_{2s}^2 = \left(\frac{\pi/2+\alpha}{\lambda}\right)^2. \tag{2.21}$$

Thus the problem that must be solved is the combination of the two integral equations given by (2.19)–(2.20) and the interface condition (2.21) along the unknown interface, $y = \eta(x)$.

3. Numerical considerations

As the location of the interface is unknown, and the dependence of the interface condition upon the velocity is quadratic, this is a highly nonlinear problem. In order to solve this problem we must resort to a numerical scheme. There is an additional complication caused by the presence of the singular point representing the sink flow being on the interface itself, which makes the problem numerically unstable near to the sink if it is not treated very carefully. An algorithm which was found to be successful is described below.

(i) Make a guess for $\eta'(s)$, the rate of change of η with respect to the arclength, at a set of points along the interface, $s_k, k = 1, 2, \ldots, N$, and fix the entry angle of the interface into the sink α, and the value of Q. A grid spacing corresponding to $s_k = s_N[(k-1)/(N-1)]^\beta, k = 1, \ldots, N$, was chosen. Choosing $\beta = 2$ was found to give accurate solutions for cases in which the interface levelled off a long way from the sink, crowding points in the area of greatest interest near to the sink. However, when waves were present on the interface, $\beta = 1$ gave much better convergence.

(ii) This guess for $\eta'(s)$ can be integrated to give $\eta(s)$, and noting that

$$x'(s) = \sqrt{1 - \eta'(s)^2}, \tag{3.1}$$

$x(s)$ can be also be obtained by integration. A trapezoidal rule integration scheme was found to be adequate for these calculations.

(iii) Using the guesses for $x, \eta, x'(s)$ and $\eta'(s)$ along the interface, the error in the integral equations (2.19)–(2.20) for A_1 and A_2 can be obtained using a guess for the values of A_1, A_2 at the same set of points, i.e. $s_k, k = 2, \ldots, N$. The error in equation (2.21) can also be evaluated. Thus by guessing $\eta'(s), A_1(s), A_2(s)$ we have a set of $3N - 3$ equations for $3N - 3$ unknowns, and can use a Newton iteration scheme to obtain a solution. Note that the values of $\eta'(0) = \arcsin \alpha, A_1(0)$ and $A_2(0)$ are known, and hence the equation was not enforced at the point $k = 1$, i.e. right at the sink.

The details of the above involve several important factors which must be considered. The singular part of the principal value integrals was removed by noting that

$$⨍_0^{z_N} \frac{\chi_j(z)}{z - z_0}\, \mathrm{d}z = \int_0^{z_N} \frac{\chi_j(z) - \chi_j(z_0)}{z - z_0}\, \mathrm{d}z + \chi_j(z_0) \ln \left[\frac{z_N - z_0}{z_0}\right], \tag{3.2}$$

where z_N corresponds to the point at which the integral is truncated. It was found that provided the truncation point was chosen to be greater than $s_N \approx 40$, depending on α, the differences in the results were minimal. Also, it is essential to include an approximation to the portion of the integral which is neglected, i.e. $s_N < s < \infty$. This correction term was found to be very important in the success of the method.

In computing the interface condition it was found that the singular terms as $x \to 0$ had to be handled very carefully, and in fact the terms of order $1/x$, which become of order $1/x^2$ in equation (2.21), were carefully eliminated manually before programming (components from the upper and lower fluid velocities cancel exactly).

It was found that if Q and α were given, the iteration scheme converged rapidly, and that λ, the upstream interface height, was obtained as part of the solution. With $N = 200$, and the truncation point at around $s_N = 40$, solutions converged to at least two-figure accuracy. The least accurate solutions occurred as α increased toward $\pi/2$, and also as the value of F_B decreased toward one. No difficulty was found in computing solutions over a wide range of values of both F_B and α except at the extremes of α. Using bootstrapping, solutions were obtained for values of α close to $\pi/2$, i.e. very close to the single-layer flow, but as α decreased, it was not possible to

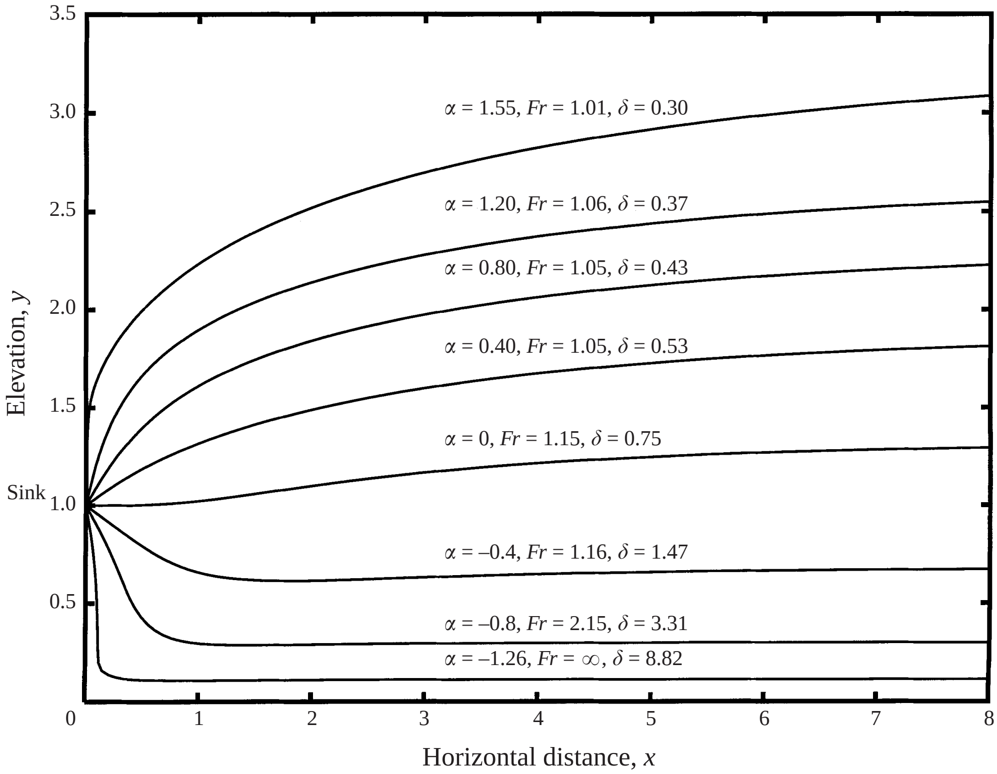

FIGURE 3. Plot of interface shapes for a range of values of the entry angle (α). All are at Froude numbers near to the minimum computed for that value of entry angle.

compute solutions beyond about $\alpha = -1.3$. Despite numerous attempts, no solutions were found with the sink very close to the bottom of the channel.

4. Results

Computations were performed over a wide range of values of α and Q. Typical solutions with different entry angles are shown in figure 3. This gives an idea of the range of interface shapes obtained when no waves were found in the solution. For values of α close to $\pi/2$ solutions are close to a single-layer flow, and look very much like the cusp solutions described earlier. As the angle decreases, the solutions are monotonic until the angle gets close to zero, when a slight inflection develops in the interface. Close to $\alpha = -\pi/2$, the surface dips down before levelling off to the correct level. There is very little surprising in these shapes. However, it appears that as $\alpha \to -\pi/2$ there is a turning point in the relative sink height, i.e. the level of the interface begins to rise again after dropping to a minimum.

One group of solutions is the case $F_B > 1$. Linear theory predicts that solutions with $F_B > 1$ cannot have waves on the interface, while those for $F_B < 1$ can. This group can be broken into two separate regions. The first is that for which δ is greater than unity, and the other where δ is less than unity. These correspond to the upstream level of the interface being greater than the height of the sink, and less than the height of the sink, respectively. In the region where the sink is below the steady level of the interface, the value of entry angle α is mainly positive and almost unaffected by the

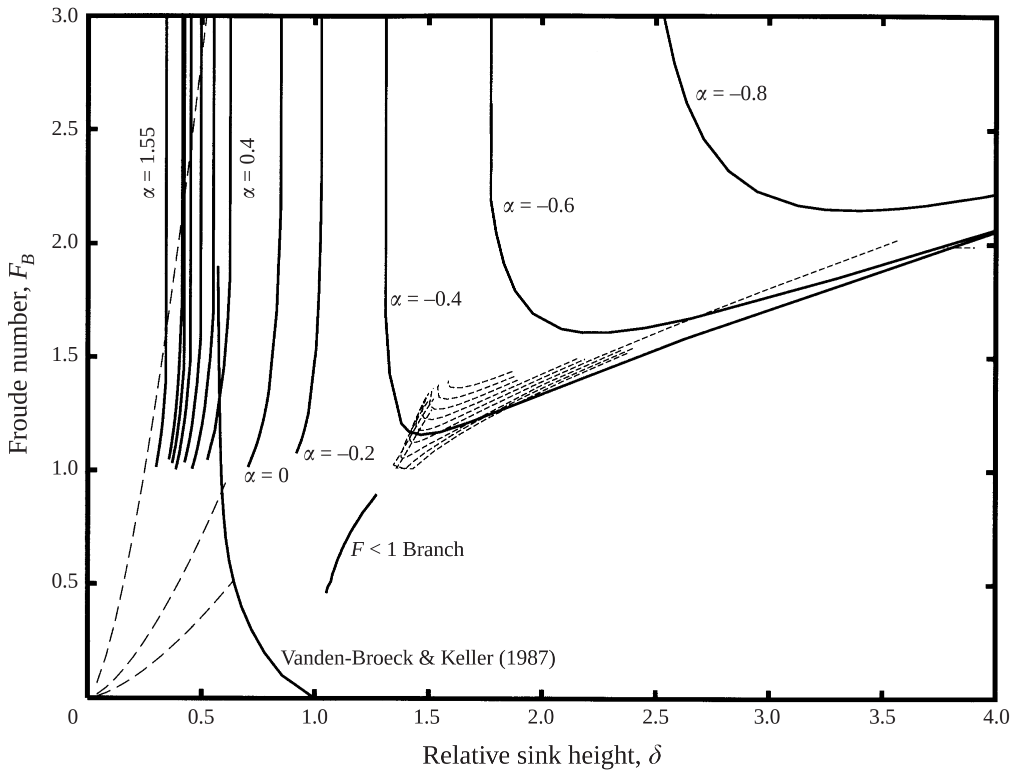

FIGURE 4. Plot of Froude number (F_B) vs. relative sink height (δ) for a range of entry angles. Includes the branch of solutions with $F_B < 1$ for which α varies, and the cusp solution branch of Vanden-Broeck & Keller (1987).

Froude number. Solutions for a fixed value of α exist for almost all values of F_B, and in almost every case, this happens over a narrow range of the sink height δ (see figure 4).

When $\delta > 1$, the values of α are predominantly negative, and for fixed values of α, the angle contours decrease, then turn about and increase again as δ increases. Thus for any fixed values of α and F_B there is a non-uniqueness in the solutions. A plot of an example of such non-unique solutions is given in figure 5. The two solutions have a different upstream level, and it is possible that one of them is stable to small perturbations and the other is not, but without a stability analysis it is difficult to say which. The minimum value for the Froude number in all of these cases is greater than one.

In the cases with $\delta > 1$ it seems reasonable to expect that there might be a set of solutions at some value of α for which F_B decreases until it hits $F_B = 1$ and then stops. In searching for these solutions an apparently new branch of solutions was found which behaved similarly to the others, but some of which decreased until they reached $F_B = 1$, then picked up again at a different value of δ and rose again, while others turned upward before reaching $F_B = 1$. These are shown on figure 4 as a cluster of dashed lines around $\delta = 1.5$, $F_B > 1$. Each dashed line corresponds to a different value of entry angle, α, and these were not investigated beyond the lines shown. It was possible to compute outward as far as one might wish as δ increased, but as δ decreased all of these curves curved upward and then terminated,

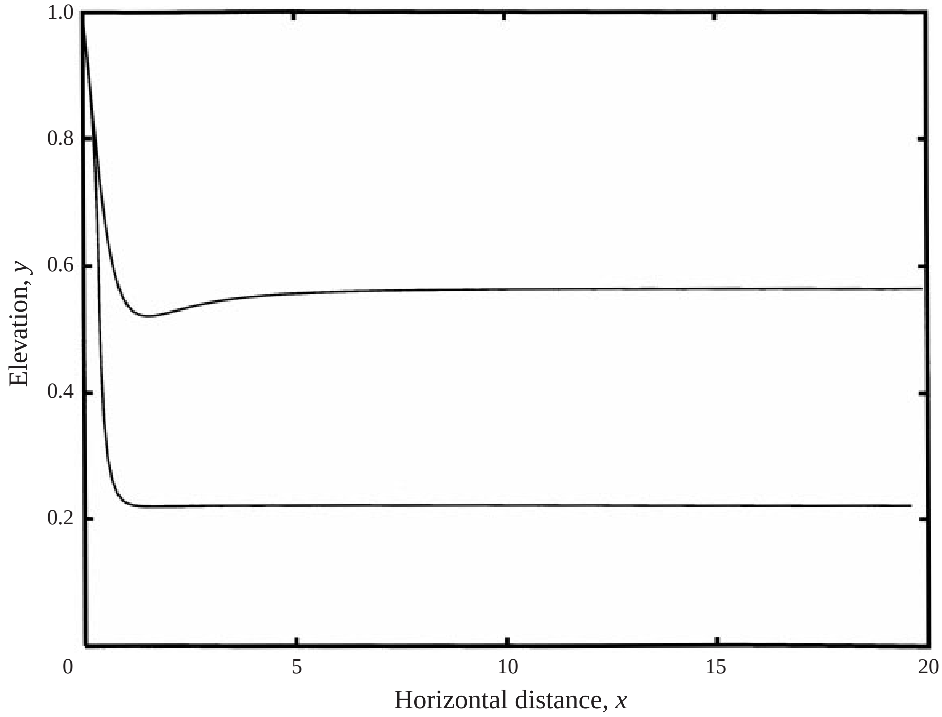

FIGURE 5. Two solutions with identical values of $F_B = 2.2$ and $\alpha = -0.6$. The upper level surface has $\delta = 1.77$ and the lower level $\delta = 4.52$.

and it was not possible to go beyond this point. The longest line here corresponds to $\alpha = 0$, and those higher to values $\alpha = 0.04, 0.08, 0.12, 0.16, 0.20$ while those below to $\alpha = -0.04, -0.08, -0.12$. Of particular interest are the curves for $\alpha = -0.08, -0.12$, since these actually terminate at $F_B \approx 1$ and then pick up again at a smaller value of δ and increase up from $F_B = 1$. These two values look as if they should have a sub-critical, $F_B < 1$, component to join the contours. No such components with $F_B < 1$ were found.

However, a branch of sub-critical solutions was found. They are shown in figure 4 as the curve labelled '$F < 1$ Branch'. These contained waves of very small amplitude that decrease as the number of discretization points increases, suggesting they are in fact waveless. They appear to be unique, i.e. only one at each value of α. In order to obtain them, the value of the parameter Q was fixed, and the angle of entry α and upstream depth λ were the outputs of the numerical procedure. An extra equation was obtained by enforcing the condition that $\eta'(s) = 0$ at the truncation point, i.e. demanding that the interface be horizontal at this point. An interface profile computed from one of these solutions is shown in figure 6, for the case $\alpha = -0.068, \delta = 1.163$, and $F_B = 0.75$.

Another group of sub-critical solutions was found with waves of varying amplitude, but these are not shown on figure 4 because of a lack of confidence about their existence. They appear to exist for a finite range of $\delta > 1$, $F_B < 1$ and $\alpha < 0$ in each case. In all cases they were for values of α close to zero and negative. Short contours for $\alpha = 0, -0.04, -0.08, -0.12$ were found. Attempts were made to compute others without success. The convergence of Newton's method to these solutions was slow, as was the convergence to an accurate solution for increasing numbers of discretization points. Any number of wavelengths outward could be computed and

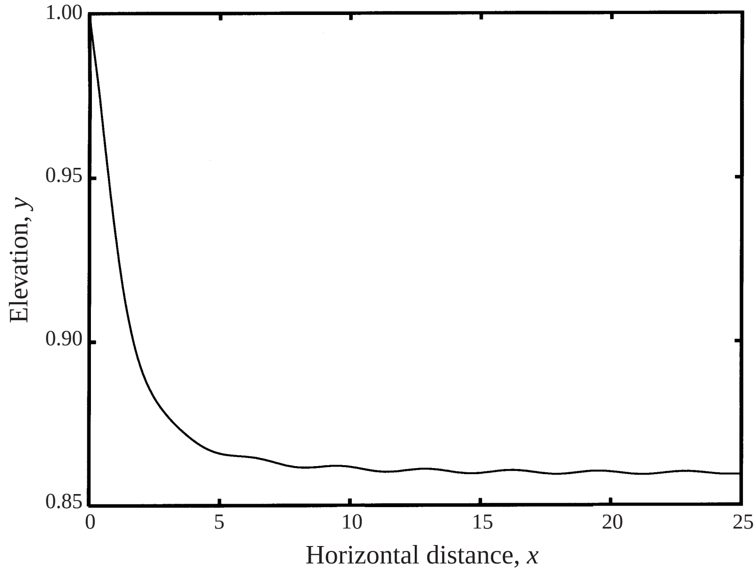

FIGURE 6. Typical interface shape for the branch of solutions with $F_B < 1$. These solutions occur at a unique F_B for each α; $F_B = 0.75$, $\alpha = -0.068$, $\delta = 1.163$ in this case. As more accurate solutions are obtained the small waves disappear.

the wave steepness appeared to increase as the ends of the contours were approached, giving a physical reason for the breakdown. However, until more accurate or more powerful solution techniques become available, the existence of these solutions must remain uncertain.

All of these solutions with $F_B < 1$ are characterized by relatively small angle of entry into the sink. The convergence of all of the sub-critical solutions was much worse than those for Froude number bigger than unity and those with waves were by far the worst. There is a strong similarity between this work and that on the case of two-layer flow in a duct presented in Forbes & Hocking (1998).

The only region in figure 4 that we are yet to discuss is the region with $F_B < 1$ and $\delta < 1$, corresponding to slow flows where the upstream level of the interface is greater than the sink depth. A number of single-layer flows are known to exist in this region involving either a stagnation point on the interface above the sink or a cusp shape at that location. Despite many attempts, no fully two-layer solutions were obtained in this region. The closest to two-layer flows that are known to exist are those of Vanden-Broeck & Keller (1987) who obtained single-layer cusped interface flows along a single curve (shown in figure 4), and Hocking & Vanden-Broeck (1998), who found that there were similar solutions with waves on a narrow band to either side of this curve (not shown). In these solutions, the interface curves downward, but does not reach all of the way down to the sink, and thus only the lower layer is flowing out. Hocking (1995) showed that in the case of a single fluid of infinite depth, these solutions appear to be a precursor to a full two-layer flow.

5. Discussion

By far the most significant results in this paper are those presented in figure 4, and in particular those for $\delta < 1$, since these correspond to the case in which the interface is above the sink, and thus represent the situation as the level falls in reservoirs during withdrawal. The real surprise is the relationship between the parameter values at which these solutions occur. In the earlier discussion it was suggested that for a given sink depth and increasing flow rate, there would be a single-layer flow until some critical Froude number, at which the drawdown would occur. At higher flow rates, we would then expect to see a decreasing angle of entry into the sink. Thus, if we were to plot contours of constant entry angle on a set of axes with relative sink height on the horizontal axis, and Froude number on the vertical axis, we would expect to see a set of roughly horizontal curves, or at least a set of curves that had every angle for every value of the sink depth. Figure 4 shows such a set of contours. It is noticeable that these lines of constant entry angle are roughly vertical if $\delta < 1$ and that for each sink depth there appears to be only a small range of valid entry angles for the steady two-layer solutions! (There are a number of single-layer solutions in this parameter domain, but we will discuss these later.) In most cases the contours terminate at a value of the upstream Froude number F_B equal to unity.

Further, as one moves down one of these constant-angle curves, there is almost no variation in the shape of the interface. The extra solid line on figure 4 shows the single-layer cusp solutions obtained by Vanden-Broeck & Keller (1987). As mentioned above, no two-layer solutions were computed for values of δ less than about 0.3, i.e. there do not appear to be solutions of this type if the sink is very close to the bottom of the lower fluid domain.

These results are not at all what might be expected, so this section of the paper is devoted to an attempt to explain them. One way to interpret the results is to consider an experiment in which water is drawn from a slot in the end of a long tank. In such an experiment, there is only a slow change in the interface level, and the flow can be thought of as quasi-steady. The results described above are somewhat confusing if one thinks about increasing the flow rate for a fixed upstream depth. That would correspond to moving upward along a vertical line in figure 4. However, for a specific sink height, the level upstream drops slowly as more fluid is withdrawn, so that the relative sink height is increasing. Assuming that the actual flux of fluid out through the slot is roughly constant, it is possible to show that the relative change in F_B vs. δ is given by

$$F_B = F_0 \left(\frac{\delta}{\delta_0} \right)^{3/2},$$

where F_0 and δ_0 correspond to the starting values in the experiment. Therefore, as the experiment proceeds, the parameters will curve upward until they eventually hit the line given as $\alpha \approx \pi/2$, when drawdown would occur.

Some examples of these curves are shown in figure 4 as the long-dashed lines starting at $\delta \approx 0, F \approx 0$. The three curves correspond to values of $F_0/\delta_0^{3/2}$ of 1, 1.5 and 8, respectively, going from the lowest to the highest. The initial value of δ_0 may be quite small so these numbers are not unrealistic. As the level drops the situation moves up the curves. If $F_B > 1$ at the drawdown point, as in the highest example, then the flow situation would move further to the right and go through all of the angles. In turn, this suggests that for each experimental situation it is the ratio of depth of the sink to total depth which is the trigger for the drawdown, rather than the Froude number. The critical drawdown Froude number in that case would actually depend on the initial Froude number and may explain the large scatter in experimental results.

If the Froude number is less than one, the only known single-layer cusp solutions are those of Vanden-Broeck & Keller (1987), shown in figure 4, and the band of wavy solutions around them obtained by Hocking & Vanden-Broeck (1998), not shown. It is possible that in that case the level drops until this particular curve is reached, as in the lower two dashed curves, when drawdown occurs. This would mean that the critical sink height at drawdown is around $\delta = 0.38$ when $F_B > 1$, and $\delta > 0.57$, depending on the initial Froude number, when $F_B < 1$. This value rises to almost $\delta = 1$ as the initial Froude number decreases. Since most experimental situations would occur at $F_B < 1$, the latter value, i.e. that given by the curve of Vanden-Broeck & Keller, is the one that would be expected.

It is also interesting to note that single-layer cusp solutions exist for almost all configurations with Froude number greater than one (see Vanden-Broeck & Keller 1987; Hocking 1991*a*), and this means that at the very least there is a non-uniqueness in the possible solutions in this region of the parameter space. It could be argued that the solutions computed in this paper are unstable and that the curve of Vanden-Broeck & Keller is the critical curve for all cases. This would avoid the ambiguity here if the 'experimental curve' described above crosses from $F_B < 1$ to $F_B > 1$ during the experiment. Another possibility is that the cusp solutions correspond to a source-like flow and the two-layer flows to a sink-type flow. It is very difficult to resolve these issues without further work, or perhaps more analysis of past experimental results.

The two-layer solutions with Froude number less than one and $\delta > 1$ presented here appear to have no relation to this argument. Experimental work would need to be used to verify or contradict these suggestions, but of the existing experimental work, only that of Gariel (1949) describes experiments for the configuration used in this paper, and there is not enough information in that paper to draw any definite conclusions. In addition, the critical drawdown in Gariel's work occurs at a Froude number of less than one, a case where no full two-layer solutions have been found.

There is one very important question to which this work does not provide an answer. What happens if the sink is actually situated on the bottom of the tank? Most of the experimental work in the literature for two-dimensional withdrawal has a configuration of this sort, although much of it has a very diffuse interface (Jirka 1979), or a different geometry near to the sink (Wood & Lai 1972). This exact geometry (with the sink on the bottom) is found in the work of Hocking (1991*b*). In this case, the relative sink depth trigger cannot occur, and consequently it must be treated as a separate case, in which there may be an absolute depth trigger. The scatter in the experimental work, in addition to the different geometries employed, do not allow us to speculate on a solution to this dilemma. Many numerical experiments have been conducted using a modified numerical scheme in an attempt to find solutions with the sink on the bottom, but although some isolated solutions have been found, these do not appear to have any pattern and hence further work must be conducted.

This paper describes a method that has been used to compute accurate numerical solutions to the problem of supercritical withdrawal through a line sink from a two-layer fluid, where the lower layer is of finite depth. The work suggests that the drawdown trigger is the relative depth of the sink, rather than the Froude number, but that it does depend on the initial configuration of the flow field. If this is correct, it means a major re-evaluation of the way we think about these problems.

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